Classical Greek Geometry Problems

Solved by means of GeoGebra

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  - Duplication of Cube
  - Pappus Problem
  - Trisection of an Angle
- Conclusions

Motivation of this work

- Learn about the math problem solving techniques from their inventors (the Greek geometers);
- Learn about the roots of Analytic Geometry;
- Learn about the origins of conics.

GREEK PROBLEM SOLVING METHODS

One of the most important achievements of early Greek geometers was the development of basic problem solving procedures:

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- Synthesis;
- Divide and Conquer;
- Neusis

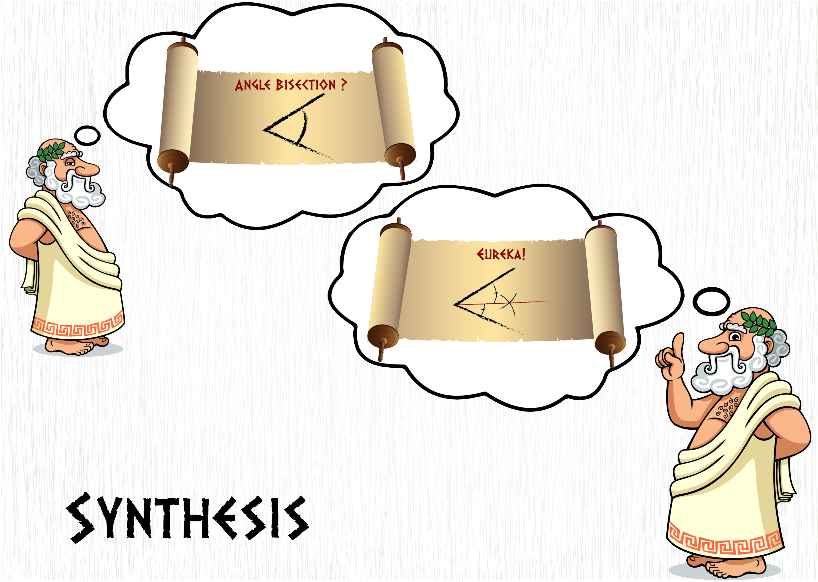
Analysis

They assumed the problem to have been solved and then, by investigating the properties of this solution, **worked back** to find an equivalent problem that could be solved on the basis of the given quantities.

UNKNOWNS
·X
·Y
GIVEN QUANTITIES
·A ·B
·C ·D
SOLUTION PROPERTIES
AX+Y=B
CX+Y=D
X= (B-D)/(A-C)
Y= (AD-BC)/(A-C)
ÀRNAPÁLIN LYŚIN

Synthesis

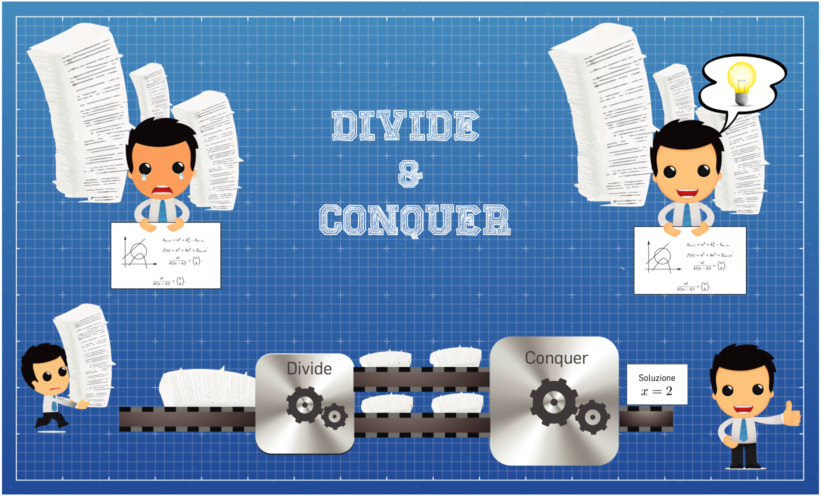
Synthesis start with a previous accepted axioms or proved theorems and proceeds therefrom to proof of a new theorem. The synthesis consists of introducing the curves (for Euclid circles and straight lines), finding their intersection, and showing that this solves the problem.


ANGLE BISECTION ?
EUREKA!
SYNTHESIS

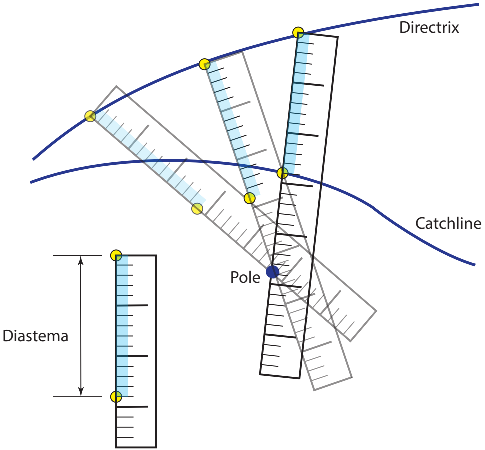
Divide and Conquer

Divide an Conquer method recursively breaks down a problem into two or more sub-problems of the same or relate type, until these become simple enough to be solved directly. The solutions to the sub-problems are then combined to give a solution to the initial problem.

Divide and Conquer



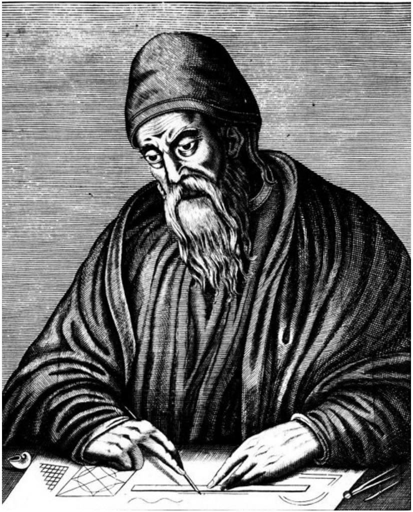
Neusis (Verging)



The $\nu\varepsilon\upsilon\sigma\iota\varsigma$ (Neusis or moving Geometry) makes use of *compass* and a marked *ruler*. The constructions are obtained by moving a marked ruler or a line or even a larger group of object, until a desired effect is achieved by trial-and-error.

DRAWING TOOLS

Euclid Drawing Tools



Euclid in his *Elements* used only two instruments:

- Compass;
- Straightedge or a ruler without markings.

Difficulties for Greek Geometers

- They had only integer numbers, no zero, no negative numbers;
- They did not use a positional system with units, tens , hundreds ...

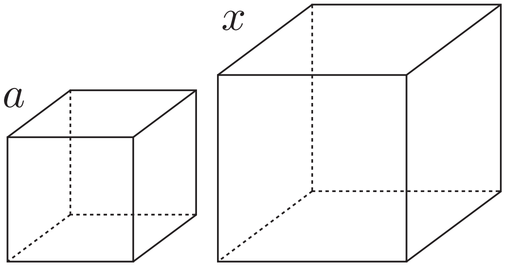
Therefore Greeks were used to **solve problems graphically**, by drawing shapes instead of using arithmetic. e.g. Division of a segment into two equal parts.

CLASSIC GREEK GEOMETRY PROBLEMS

Duplication of Cube

The citizens of Delos consulted the oracle at Delphi in order to learn how to defeat a plague. The oracle responded that they must double the size of the altar to Apollo, which was a regular cube. The Delians consulted Plato, who was able to interpret the oracle as the mathematical problem of doubling the volume of a given cube.

Given a cube of side a find the side x of new cube with volume $x^3 = 2a^3$.



$$x^3 = 2a^3$$

The Maenecmus' solution from Apollonius' Treatise on Conic Sections

Let a and b be two segments of given size and x and y two proportional quantities. Such that:

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b} \; . \tag{1}$$

From equation (1) follows:

$$\frac{a^3}{x^3} = \frac{a}{b} \tag{2}$$

We can switch from equation (1) to equation (2) taking into account that:

$$\frac{a^3}{x^3} = \frac{a^2}{xy} = \frac{ay}{xb} \tag{3a}$$

$$ab = xy \tag{3b}$$

$$\frac{a^3}{x^3} = \frac{a}{b} \tag{3c}$$

Without loss of generality, we let

$$\frac{a}{b} = \frac{1}{2} \ .$$

Hence, from (2) the following cubic equation is obtained:

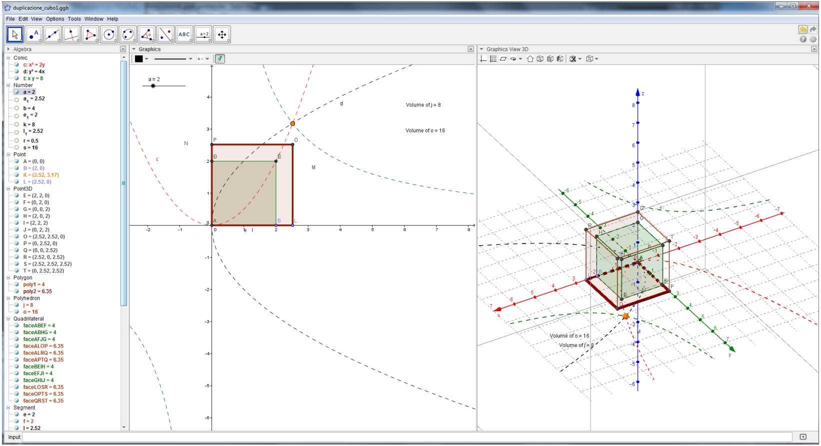
$$x^3 = 2a^3 \ . \tag{4}$$

From equation (1) we can deduce other two equations:

$$x^2 = ay \tag{5a}$$

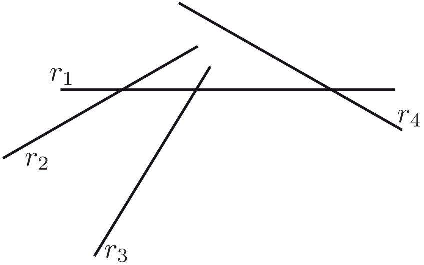
$$y^2 = bx \tag{5b}$$

GeoGebra Solution

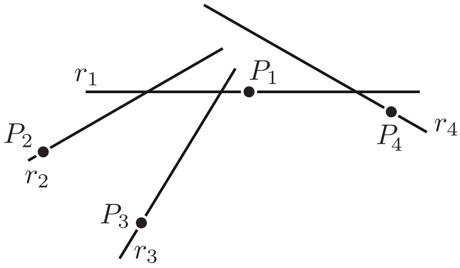


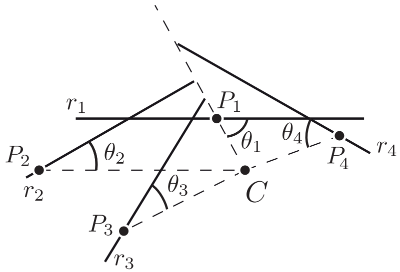
Pappus Problem

Let r_1, r_2, r_3, r_4 be straight lines arbitrarily oriented.



Let P_1, P_2, P_3, P_4 be distinct points on each of the previous straight lines.





Assign arbitrarily four angles θ_1, θ_2, θ_3e θ_4.
Find the locus of points C such that:

- CP_1, CP_2, CP_3 and CP_4 form with r_1, r_2, r_3, r_4, the angles θ_1,θ_2, θ_3 and θ_4, respectively;
- is fulfilled the condition:

$$\frac{CP_1 \cdot CP_2}{CP_3 \cdot CP_4} = \text{Constant}$$

History of the Problem

The problem was first stated in the 7^{th} book of Pappus *Collection* (c. A.D. 340).

> *Though, as I said these proposition follow the locus on four lines, geometers have by no means solved them to the extent that the curve can be recognized.*

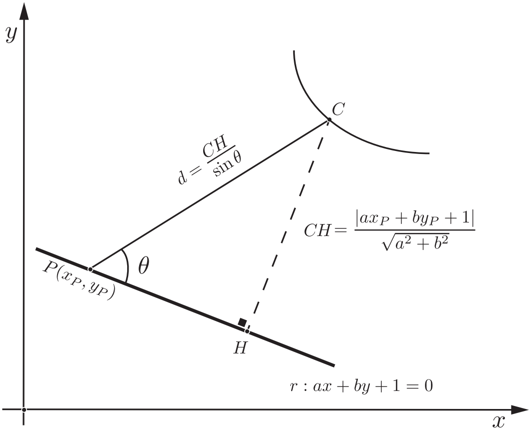
Many centuries later, *Golius*, brought Pappus Problem to Descartes attention. Descartes used this problem to test his own problem solving method (a mix of Greek analysis and symbolic notation introduced by Viète).

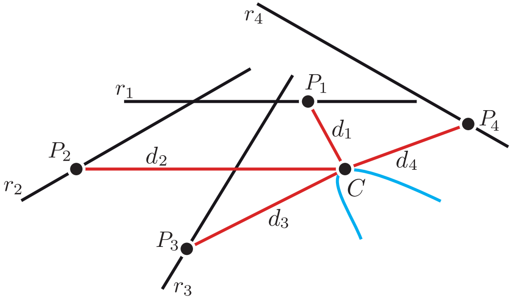
Importance

The solution by Descartes is the **first** application of analytic geometry, a topic that all high-school students must deal with.

A Modern Solution

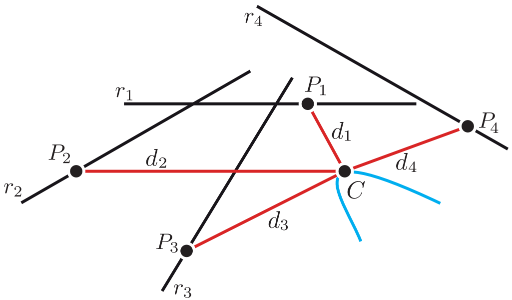
The distance d between C and a generic point P_i on straight lines r_i is computed through the following scheme:





The Geometry Locus of point C is given by

$$CP_1 \cdot CP_2 - CP_3 \cdot CP_4 = 0 \tag{6}$$



The Geometry Locus of point C is given by

$$CP_1 \cdot CP_2 - CP_3 \cdot CP_4 = 0 \tag{6}$$

since, $CP_i = d_i \ (i = 1, \ldots, 4)$,

$$d_1 \cdot d_2 - d_3 \cdot d_4 = 0$$

wxMaxima 13.04.2 [pappo.wxm*]

File Edit Cell Maxima Equations Algebra Calculus Simplify Plot Numeric Help

Pappus Problem - 4 straight lines

0.1 Problem Statement:

Given 4 arbitrary lines(r1, r2, r3, r4),4
arbitrary angles, find the locus of points C such that:
- CP1, CP2, CP3 e CP4
form with r1, r2, r3, r4,
the angles theta1,theta2, theta3 and theta4,
respectively;
- is fulfilled the condition:
CP1*CP2=k*(CP3*CP4)
where points P_i belongs to r_i (i=1,2,3,4).

First compute segment lenghts

```
--> kill(all);
    d1:(a1*x+b1*y+1)/(sin(theta1)*sqrt(a1^2+b1^2));
    d2:(a2*x+b2*y+1)/(sin(theta2)*sqrt(a2^2+b2^2));
    d3:(a3*x+b3*y+1)/(sin(theta3)*sqrt(a3^2+b3^2));
    d4:(a4*x+b4*y+1)/(sin(theta4)*sqrt(a4^2+b4^2));
```

0.2 Compute the locus equation

```
--> eq:expand(d1*d2-d3*d4)$
    eq:num(ratsimp(eq));
```

0.3 Extract the coefficients from equation.
Ax^2+By^2+Cxy+Dx+Ey+F=0

Term of y^2

```
--> B:ratcoef(eq,y,2);
```

Term of x^2

```
--> A:ratcoef(eq,x,2);
```

Term of x*y

```
--> u:ratcoef(eq,y,1);
    C:ratcoef(u,x,1);
```

Term of y

```
--> E:ratcoef(u,x,0);
```

Term of x

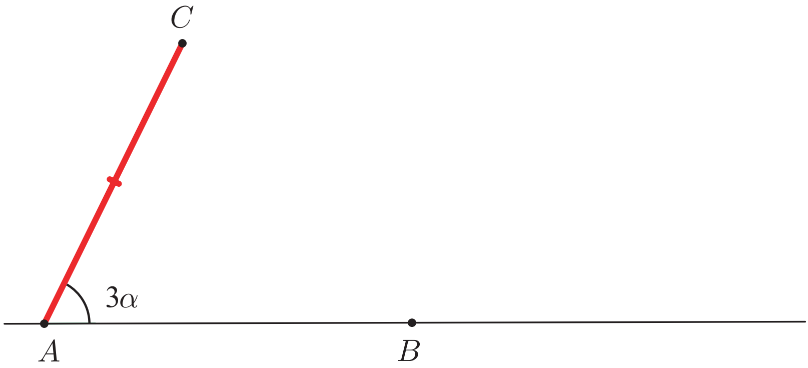
```
--> v:ratcoef(eq,x,1);
    D:ratcoef(v, y,0);
```

Known Term

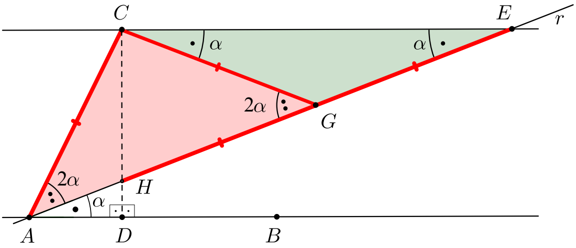
```
--> c1:ratcoef(eq,x,0);
    F:ratcoef(c1,y,0);
```

Angle Trisection

Divide the angle 3α in three equal parts.

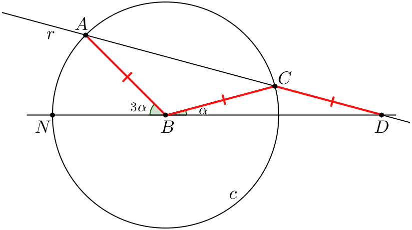


Hippocrates' Solution



- Draw a line r through A and locate intersections E and H with horizontal and vertical lines;
- G is midpoint of EH;
- Change the angular coefficient of r until $GH \approx CA$.

Archimedes' Solution



- Draw line through A and locate point C and C intersection with circle and the line through B and N;
- Change the angular coefficient of r until $BC \approx CD$.

Conclusions

- Classical Greek geometry problems have been reviewed.
- GeoGebra worksheets have been produced.