Simulation of Curve Tracing Linkages
by means of GeoGebra

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Summary of this presentation

- Motivation of the work
- An early mathematical linkage (Cube duplication)
- Simulation of linkages with GeoGebra

Motivation

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- The understanding of mathematical linkages working principles shapes up mathematical skills and shed further light on many math methods.
- GeoGebra allows the modeling of these linkages in a parametric way.

What is it a mathematical linkage ?

A mathematical machine is a linkage designed and built to move or transform a point, a line segment or a plane figure in accordance with a mathematical law.

Early use of mathematical machines

The first mathematical machine have been designed by Greek geometers in order to solve problems that could not be solved with straight edge and compass.

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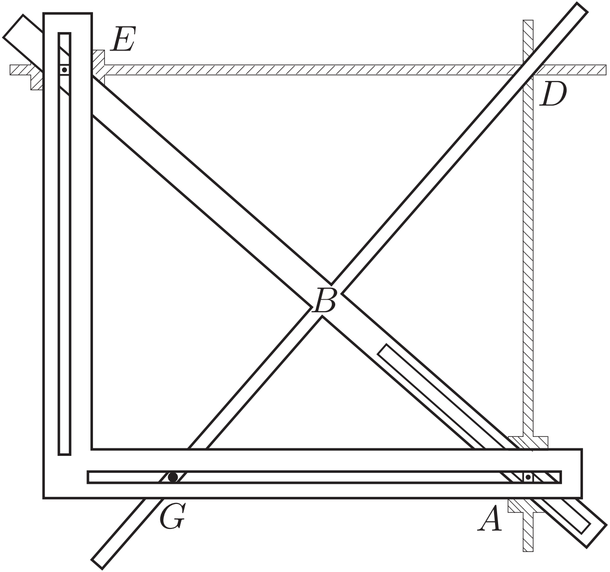
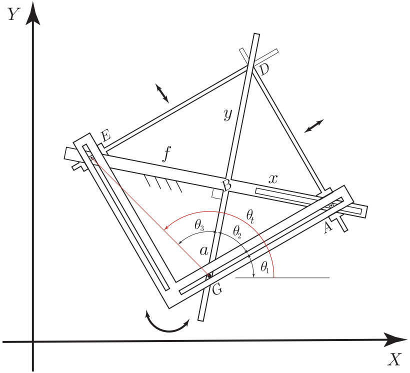
Some geometers were critical on the use of these devices.
From Plutarch:

> *Plato reproached the disciples of Eudoxus, Archytas and Menaechmus for resorting to mechanics and instrumental means for resolving the problem of cube duplication ...*

DUPLICATION OF CUBE

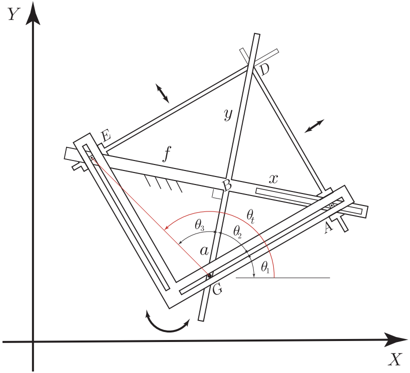
Eutocius' Solution





From triangles similitude:

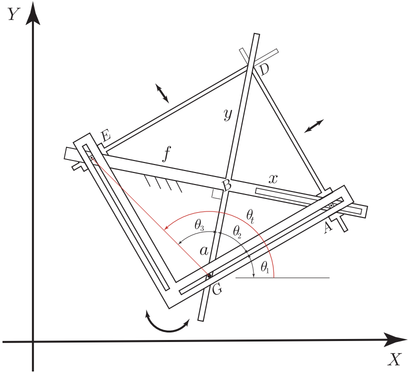
$$a : x = x : y = y : f$$



From triangles similitude:

$$a : x = x : y = y : f$$

$$\therefore\ x = \sqrt[3]{a^2 f}$$



From triangles similitude:

$$a : x = x : y = y : f$$

$$\therefore \; x = \sqrt[3]{a^2 f}$$

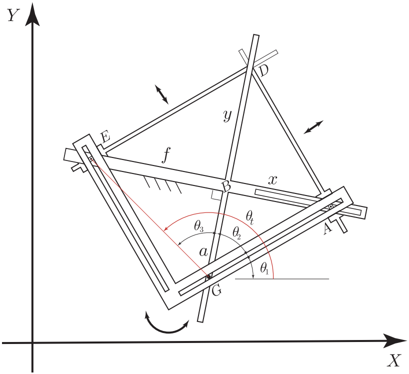
If $f = 2a$ then

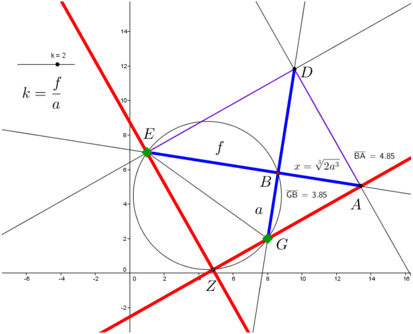
$$x = \sqrt[3]{2a^3}$$

$$\theta_t = \mathrm{ATAN2}\left(Y_E - Y_G, X_G - X_E\right) \tag{1}$$

$$\theta_3 = \tan^{-1}\frac{f}{a}\,, \quad \theta_2 = \frac{x}{a}\,, \quad \theta_1 = \theta_t - (\theta_3 + \theta_2) \tag{2}$$

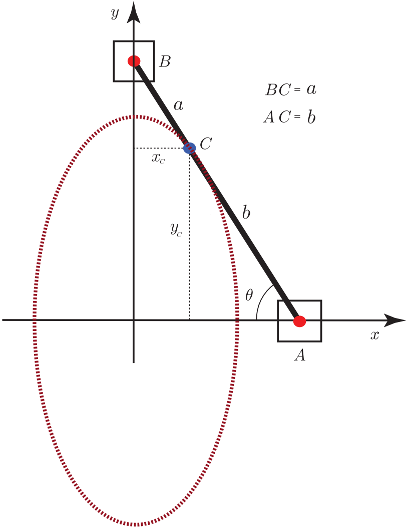
$$Y = \tan\theta_1\left(X - X_G\right) + Y_G \tag{3}$$


Y
X
E
D
f
y
x
B
A
G
a
θ_3
θ_2
θ_t
θ_1


k = 2
$k = \frac{f}{a}$
E
D
f
B
$x = \sqrt[3]{2a^3}$
BA = 4.85
GB = 3.85
a
A
G
Z

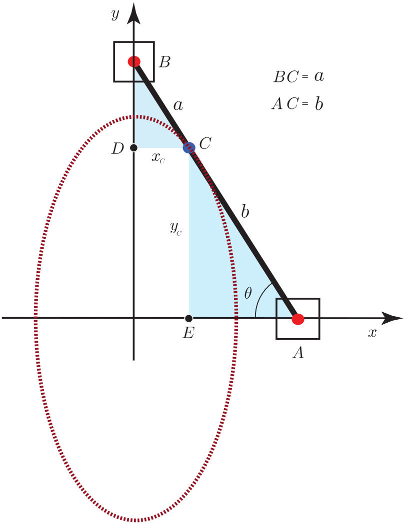
LEONARDO ELLISOGRAPH

Leonardo da Vinci Ellisograph



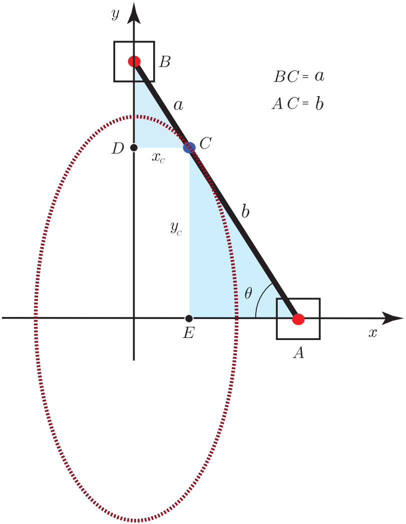
The apparatus is made of a coupler AB and two sliders, moving along orthogonal axes. The drawing tip is in C

Theory



From the figure we observe the similitute of the right angle triangles CBD and AEC
The following equalities can be established:

$$\widehat{DCB} = \widehat{EAC} = \theta \qquad (4)$$



From the figure geometry, the following equalities hold:

$$\cos\theta = \frac{DC}{BC} = \frac{x_c}{a} \qquad (5)$$

$$\sin\theta = \frac{AE}{AC} = \frac{y_c}{b} \qquad (6)$$

Substituting (5) and (6) into

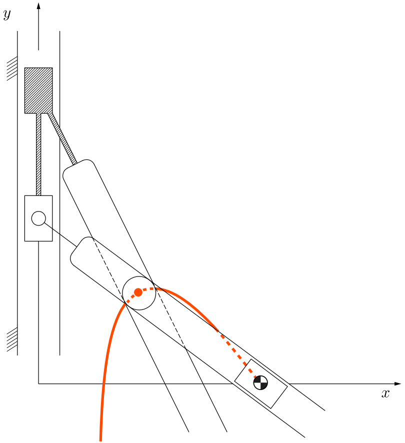
$$\cos^2\theta + \sin^2\theta = 1 \tag{7}$$

the canonical form of the ellipse, traced by point C is obtained

$$\frac{x_c^2}{a^2} + \frac{y_c^2}{b^2} = 1 \tag{8}$$

DESCARTES HYPERBOLOGRAPH

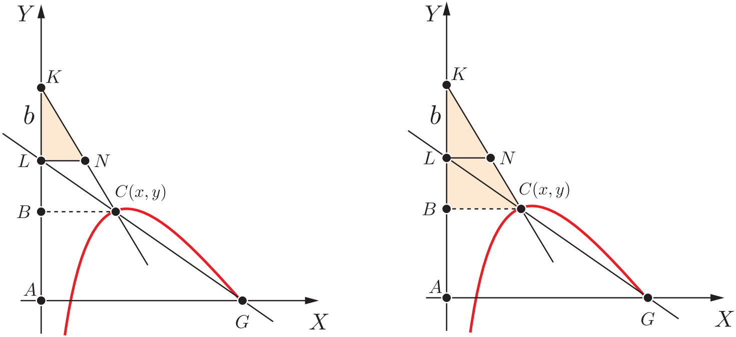
Descartes Hyperbolograph



The apparatus is composed of:

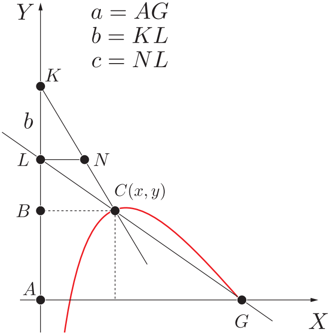
- Two vertical sliders at constant distance;
- A swinging block;
- A roller within two straight line guides

Theory



With reference to the figure, we observe the similitude of triangles NLK and CBK. Therefore, the following proportion can be established:

$$\frac{NL}{KL} = \frac{x}{BK} \tag{9}$$



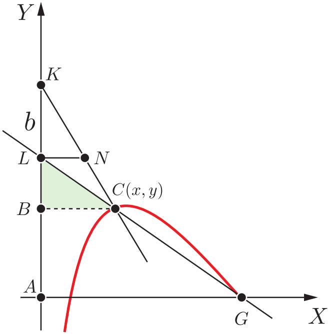
whence,

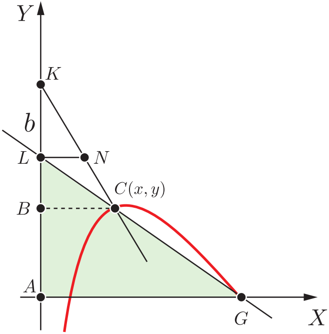
$$BK = \frac{b}{c}x \ . \quad (10)$$

With reference to the figure follows:

$$BL = BK - LK = \frac{b}{c}x - b \quad (11)$$

$$AL = BA + BL = y + \frac{b}{c}x - b \ . \quad (12)$$





We observe also a second similitude of triangles LBC and LAG, from which the following proportion can be deduced:

$$\frac{CB}{BL} = \frac{AG}{AL} \tag{13}$$

which can be rewritten in the form:

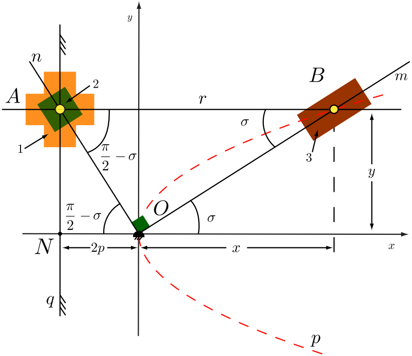
$$\frac{x}{\frac{b}{c}x - b} = \frac{a}{y + \frac{b}{c}x - b}$$

$$x^2 = x(a + c) - \frac{c}{b}xy - ac \ . \tag{14}$$

recognized as an *hyperbola*

ANTONOV PARABOLOGRAPH

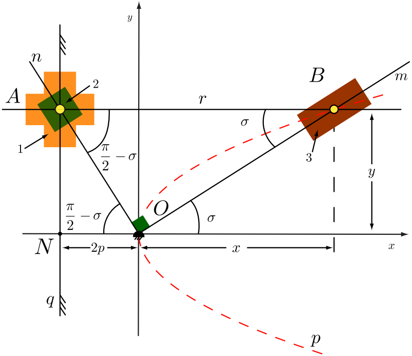
Antonov Parabolograph



The apparatus is made of:

- A right angle shaped member pinned and fixed at O whose links slide along the swinging blocks pinned in A and B ;

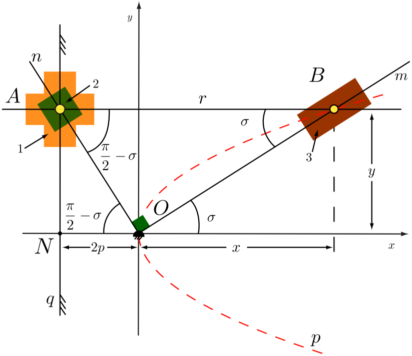
Antonov Parabolograph



The apparatus is made of:

- A right angle shaped member pinned and fixed at O whose links slide along the swinging blocks pinned in A and B ;
- point A slides along fixed straight line q;

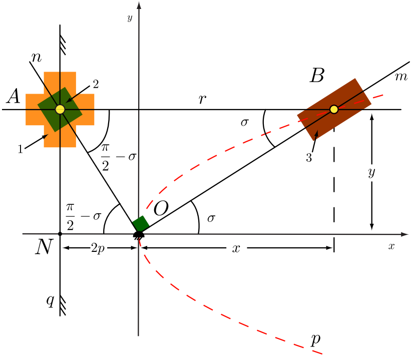
Antonov Parabolograph



The apparatus is made of:

- A right angle shaped member pinned and fixed at O whose links slide along the swinging blocks pinned in A and B ;
- point A slides along fixed straight line q;
- Point B slides along rotating straight line m amd trace parabola p.

Theory

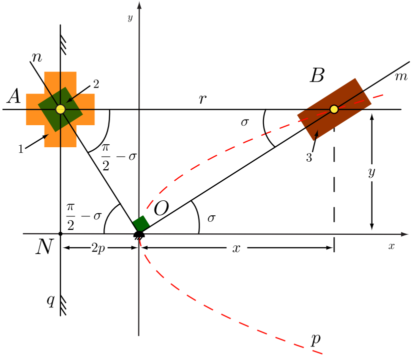


- From the figure we obtain the following equalities:

$$OA = \frac{2p}{\sin\sigma} \quad (15a)$$

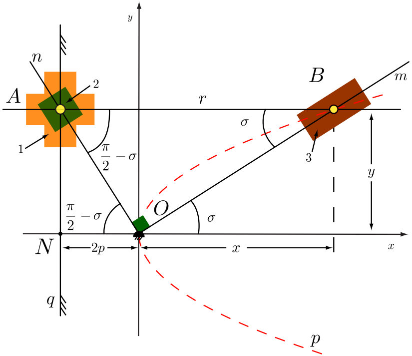
$$y_B = y_A = OA\cos\sigma \quad (15b)$$

$$OB = OA\frac{\cos\sigma}{\sin\sigma} \quad (15c)$$



Substituting (15a) into (15b) we obtain:

$$y_B = \frac{2p}{\tan\sigma}\ . \qquad (16a)$$



Substituting (15a) into (15b) we obtain:

$$y_B = \frac{2p}{\tan\sigma}\ . \qquad (16a)$$

The abscissa x_B of B is:

$$x_B = OB\cos\sigma \qquad (17)$$

Substituting (15a) and (15b) into (17) follows:

$$x_B = \frac{2p}{\tan^2\sigma} \qquad (18)$$

From (16a) we obtain:

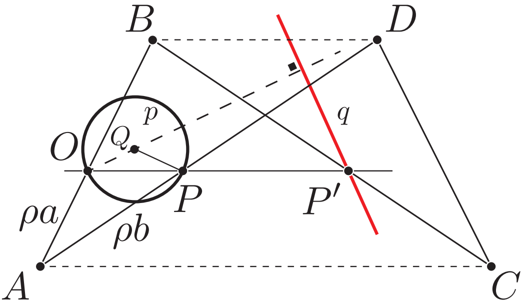
$$\tan\sigma = \frac{2p}{y_B} \tag{19}$$

Substituting (19) into (18) follows:

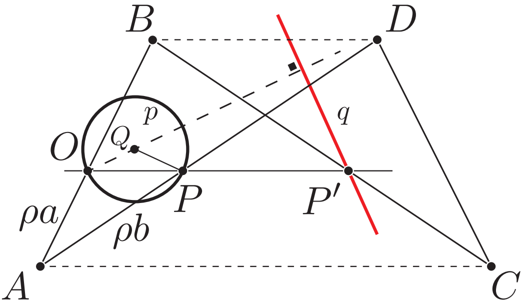
$$x_B = \frac{y_B^2}{2p} \tag{20}$$

HART INVERSOR

Hart Inversor



The apparatus is made of a crossed parallelogram $ABCD$ and a crank QP, which drives point P along a circle p.



- Point O and Q are fixed and pinned.
- Point P' is such that $OP \cdot OP'$=const.

While point P traces the circle p, point P' slides along straight line $q \perp OQ$

CONCLUSION

Mathematical linkages reproduced

1. Eutocius linkage for cube duplication
2. Leonardo da Vinci Ellipsograph
3. Descartes Hyperbolograph
4. Antonov Parabolograph
5. Hart Inversor
6. Delaunay Ellipsograph
7. Peaucellier Inversor

Conclusions

- The feasibility of GeoGebra for the simulation of the mathematical linkages has been tested.

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- All the GeoGebra files produced can be downloaded from `www.geogebraitalia.org`